JOURNAL OF AIRCRAFT
Vol. 35, No. 4, July–August 1998

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Comparison of Shock Calculation Methods

Lael vonEggers Rudd* and Mark J. Lewis†
University of Maryland,
College Park, Maryland 20742-3015

Introduction

COMMONLY USED forms of the oblique shock theta–beta–Mach relationship define surface angle θ as a function of shock angle β at a given upstream Mach number M_1 and a ratio of specific heats γ; iterative methods are generally used to find shock angle as a function of surface angle. However, several versions of a closed-form solution have been derived but have not been widely used either because they are perceived to be too complicated or are simply not well known. It is of interest to determine whether these closed-form solutions, presenting shock angle as a function of surface angle and Mach number, are indeed useful and time saving, or are merely of academic interest. This work has compared shock-angle calculations using both closed-form and iterative root-solving techniques. Both the common weak shock solution and the strong shock solution have been studied. It has been found that the closed-form solutions are indeed of practical use because they enhance efficiency in computer algorithms for all but the most limited of cases.

The well-known theta–beta–Mach relation between free-stream Mach number, flow deflection angle, and the corresponding shock wave angle is given by

$$\frac{\tan(\beta - \theta)}{\tan\beta} = \frac{(\gamma - 1)M_1^2 \sin^2\beta + 2}{(\gamma + 1)M_1^2 \sin^2\beta} \tag{1}$$

Two closed-form solutions are used in this paper. The first solution states that[1]

$$\tan\beta = \frac{b + 9a\tan\mu}{2(1 - 3ab)} - \frac{d(27a^2\tan\mu + 9ab - 2)}{6a(1 - 3ab)} \times \tan\left[\frac{n}{3}\pi + \frac{1}{3}\arctan\frac{1}{d}\right] \tag{2}$$

where n can be 0, 1, -1 corresponding to the weak shock solution, strong shock solution, and a physically meaningless solution, respectively, μ being the Mach wave angle, and

$$a = \left(\frac{\gamma - 1}{2} + \frac{\gamma + 1}{2}\tan^2\mu\right)\tan\theta$$

$$b = \left(\frac{\gamma + 1}{2} + \frac{\gamma + 3}{2}\tan^2\mu\right)\tan\theta$$

$$d = \sqrt{\frac{4(1 - 3ab)^3}{(27a^2c + 9ab - 2)^2} - 1}$$

A second form of the solution was shown by Mascitti[2]

$$\sin^2\beta = (b/3) + \tfrac{2}{3}(b^2 - 3c)^{1/2}\cos[(\Phi + n\pi)/3] \tag{3}$$

where n can be 0, 2, 4 corresponding to the strong shock solution, a physically meaningless solution, and the weak shock solution, respectively, and

$$\cos\Phi = [\tfrac{9}{2}bc - b^3 - \tfrac{27}{2}(-\cos\theta/M_1^2)]/(b^2 - 3c)^{3/2} \tag{4a}$$

$$b = -\left(\frac{M_1^2 + 2}{M_1^2}\right) - \gamma\sin^2\theta \tag{4b}$$

$$c = (2M_1^2 + 1)/M_1^4 + [(\gamma + 1)^2/4 + (\gamma - 1)/M_1^2]\sin^2\theta \tag{4c}$$

For this work, the preceding methods are compared to both a common secant iteration method of Eq. (1), and an iteration formula for the weak and strong shock solution given by Collar and reported by Houghton and Brock.[3] Collar writes Eq. (1) as a cubic with the solution of one root, the weak shock solution, as

$$x_{n+1} = +\sqrt{(M_1^2 - 1) - \frac{[(\gamma + 1)/2]M_1^2\tan\theta}{x_n + \{[(\gamma + 1)/2]M_1^2 + 1\}\tan\theta}} \tag{5}$$

where x begins at the Mach wave angle at $n = 1$, and $x = \cot\beta$ for the weak shock solution at the final value of $n + 1$. The strong shock solution is found by taking the x value of the weak shock solution

$$[\tfrac{9}{2}bc - b^3 - \tfrac{27}{2}(-\cos\theta/M_1^2)]/(b^2 - 3c)^{3/2}$$

factoring it out of the cubic equation, and solving for the positive x root of the remaining quadratic equation. This value of x is then the cot β of the strong shock solution.

Small Angles

Both closed-form solutions suffer from problems of definition at zero and accuracy deterioration at small angles. Wellmann's[1] closed-form solution is undefined at $\theta = 0$. The solution should degenerate to the limiting case of a Mach wave at $\theta = 0$; however, the coefficients can become small enough in a computer solution, particularly when multiplied by each other, that the solution loses accuracy as θ approaches zero. This can cause divide-by-zero errors at small deflection angles.

Received Oct. 26, 1997; revision received Feb. 5, 1998; accepted for publication Feb. 12, 1998.

*Graduate Research Fellow, Department of Aerospace Engineering.
†Associate Professor, Department of Aerospace Engineering. Associate Fellow AIAA.

Table 1 Representative values of weak shock solution algorithm CPU run-times

Machine platform	Mach number	Deflection angle, deg	Wellmann,[1] μ	CPU, Mascitti,[2] μs	Time, Houghton and Brock,[3] μs	Secant, μs
Sparc5	2	5	25	58	22	83
Sparc20	2	5	16	29	12	49
DEC Alpha	2	5	2.6E−2	2.6E−2	11	18
Cray Y-MP	2	5	2.9E−3	0.26	39	15
Sparc5	10	5	23	55	33	81
Sparc20	10	5	15	28	18	48
DEC Alpha	10	5	2.6E−2	2.6E−2	18	15
Cray Y-MP	10	5	2.9E−3	0.24	39	25
Sparc5	2	20	24	58	52	120
Sparc20	2	20	15	29	27	70
DEC Alpha	2	20	2.6E−2	2.5E−2	29	25
Cray Y-MP	2	20	2.9E-3	0.26	53	40
Sparc5	10	20	24	55	49	73
Sparc20	10	20	15	29	26	43
DEC Alpha	10	20	2.6E−2	2.5E−2	29	14
Cray Y-MP	10	20	2.9E−3	0.24	34	39

Mascitti's[2] closed-form solution exhibits a similar behavior at small angles, though it is defined at a deflection angle of zero. Small angle roundoff errors in the calculation of Φ in Eq. (4) can also cause divide-by-zero errors. This problem is easily solved by increasing the denominator segment of cos Φ in Eq. (4); for double precision, an increase of 1×10^{-15} avoids divide-by-zero errors and produces negligible change in the solution value.

Even if no divide-by-zero errors occur, accuracy is lost with both solutions when solving for the weak shock solution because of large roundoff errors at θ approaching zero. The error of both solutions increases with increasing Mach number and decreasing deflection angle. Wellmann's[1] solution is more sensitive to this; for instance, at Mach 15 at a deflection angle of 1 deg Wellmann's solution for β is too high by almost one-tenth of a degree, while Mascitti's method is too high by only two-thousandths of a degree. This problem becomes unimportant when double precision variables are used.

Mascitti's solution with double precision has accuracy that is comparable to or more superior than the iterative solutions at all deflection angles. Wellmann's solution can never be defined at a deflection angle of 0 deg, but the use of double precision allows Wellmann's method to be used at deflection angles greater than five-thousandths of a degree and it has comparable accuracy to the iterative solutions and Mascitti's formula. Of course, physically, the accuracy of such small deflection angles may be irrelevant. At a deflection angle smaller than one-hundredth of a degree, there is less than a 0.1% difference between the shock angle and the Mach wave angle. The limitations of the closed-form solutions at very small deflection angles are of primarily academic interest and are provided as a warning for the computer programmer.

When solving for the strong shock solution, deflection angles less than 1 deg are less relevant because the solution is nearly a normal shock. Above 1 deg, Wellmann's solution does not suffer noticeable accuracy deterioration at small deflection angles and can be solved using single-precision accuracy. Mascitti's solution still requires double-precision accuracy when calculating the strong shock solution.

Results

Four machines have been used to calculate computation times: Sun Sparc5, Sun Sparc20, DEC Alpha 3000-300, and Cray Y-MP2E/232. Shock angles have been calculated to a four-decimal-place accuracy. Wellmann's and Mascitti's solutions have been calculated with double precision, whereas Collar's[3] method and the secant method used single precision. Algorithm run-times have been calculated at Mach numbers of 2, 5, 10, and 15 at angles of 0.01, 1.0, 2.0, 5.0, 10.0, 20.0, 30.0, and 40.0 deg, when applicable for the weak shock solution. The Mach numbers and deflection angles are the same when the strong shock solution has been calculated, with the exception that 0.01 deg was not used. Mach numbers and deflection angles have been placed in an array matrix for the different algorithms, with each algorithm having been placed in a loop and run between 10^5 and 10^6 cycles. Run-times have then been divided by the number of loop iterations to determine an average CPU run-time per calculation. All algorithms were coded as efficiently as possible in Fortran 77, without compiler optimization. Care was taken to be sure that differences in run-time did not reflect compiler optimization effects, including the use of multiple cycle counts for each algorithm from which it was confirmed that the time per calculation was independent of the number of calculations performed.

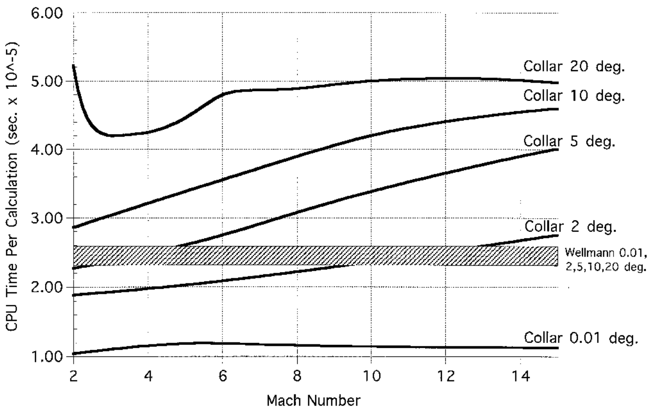

Fig. 1 Sun Sparc5 CPU calculation times of solutions at varying Mach numbers and deflection angles.[1,3]

Table 1 presents typical weak shock solution run-times on each platform for each solution method. Wellmann's[1] version is the fastest of the closed-form solutions on the Sun Sparc20 and Sun Sparc5, being an average of almost 2.5 times faster than Mascitti's[2] solution. Houghton and Brock's[3] method is the fastest of the iterative solutions, running anywhere from more than 1.5 to 6 times faster than the secant method, depending on the Mach number and deflection angle. Houghton and Brock's method run-times generally increase with deflection angle and Mach number, while Wellmann's solution run-times are relatively constant. At small angles Houghton and Brock's method is faster than Wellmann's solution. Figures 1 and 2 show the CPU times of each solution vs Mach number at various deflection angles. As can be seen the closed-form solutions are faster at deflection angles that are greater than 3–7

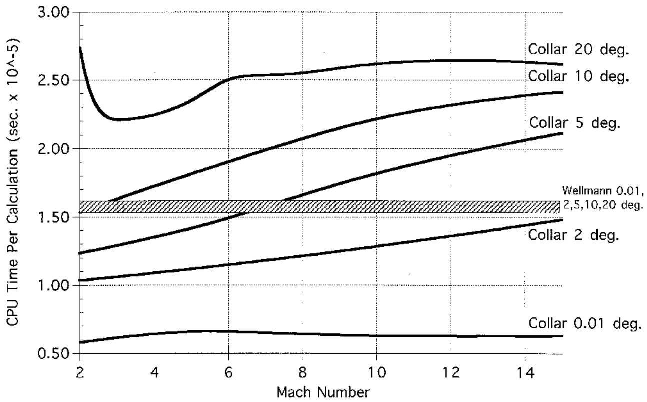

Fig. 2 Sun Sparc20 CPU calculation times of solutions at varying Mach numbers and deflection angles.[1,3]

deg, depending on the Mach number and the Sparc used. Below those angles, Houghton and Brock's method is fastest.

For the DEC Alpha, both closed-form solutions are faster by two to three orders of magnitude over iterative solutions. The closed-form solutions have about the same run-times of around 2.5×10^{-8} s/calculation, which is nearly invariant with Mach number or deflection angle. The iterative solution run-times are dependent on deflection angles and range from 4.3×10^{-6} to 3.8×10^{-5} s/calculation.

On the Cray Y-MP2E/232, closed-form solution run-times are faster, again by orders of magnitude. Run-times for Wellmann's solutions and Mascitti's are 2.9×10^{-9} and 2.4×10^{-7}, respectively, both being invariant with the Mach number and deflection angle. If arrays are not used and the Mach number and deflection angle are initially set, Mascitti's solution can have comparable run-times to Wellmann' solution on the Cray Wellmann's. This is mainly because of the uniqueness of optimization built into the Cray compiler. The iterative solutions have times ranging from 5.0×10^{-6} to 5.6×10^{-5} s/calculation, depending on the deflection angle.

The results for the strong shock solutions are similar to the weak shock solution run-times with the major exception coming from the Sparc machines. Because Wellmann's solution uses only single precision, run-times are reduced by more than half. This makes Wellmann's solution faster than Houghton and Brock's method at any deflection angle and Mach number.

Conclusions

Closed-form solutions of the theta–beta–Mach relationship prove to be faster than iterative solutions for practically all cases on the four machines examined. Note that an exception to this was found when computing the weak shock solution on the Sparc stations for some Mach numbers at deflection angles less than about 5 deg, where Houghton and Brock's method was the fastest. The primary explanation for this is in the number of trigonometric function calls required for each method.

Solving with either of the closed-form solutions requires many trigonometric function calls, which can require substantial CPU time, depending on the individual machine; however, both iterative methods used here also require trigonometric function calls as well as potentially CPU-intensive logic statements. In the secant method, trigonometric functions are used to calculate the maximum deflection angle for one of the initial guess values, and in evaluating the theta–beta–Mach relation during each iteration. Convergence typically requires about five iteration cycles. Compared to the secant method, Houghton and Brock's iteration method has a smaller number of trigonometric function calls per cycle, but may have a larger number of cycles to reach convergence. This method must perform a minimum of two iterations per solution, but the number of iterations generally increases with increasing deflection angles and Mach numbers. For instance, at Mach 10 at a deflection angle of 10 deg, 18 iterations are required to converge on the solution to within a four-decimal-place accuracy with single-precision variables.

Acknowledgment

The authors thank George Emanuel of the University of Oklahoma for bringing these solutions to their attention.

References

[1]Wellmann, J., "Vereinfachung von Rechnungen am schiefen Verdichtungsstoß," *Deutshe Luft- und Raumfahrt*, Forschungsbericht 72-11, Duestche Forschungs- und Verssuchsanstalt für Luft- und Raumfahrt, Institut für Aerodynamik, Brauschweig, Germany, Jan. 1972.

[2]Mascitti, V. R., "A Closed-Form Solution to Oblique Shock-Wave Properties," *Journal of Aircraft*, Vol. 6, 1969, p. 66.

[3]Houghton, E. L., and Brock, A. E., *Tables for Compressible Flow of Dry Air*, 3rd ed., Edward Arnold Ltd., Bungay, Suffolk, England, UK, 1988.

Empirical Model of Transport and Decay of Aircraft Wake Vortices

Lakshmi H. Kantha*
University of Colorado, Boulder, Colorado 80309-0431

Nomenclature

$a_{1,2,3,4}$ = empirical constants
b = wingspan
$c_{1,4,5,6}$ = constants in vortex decay equation
D_1 = distance between the two wake vortices
D_2 = distance between one wake vortex and the image vortex of the other
H = height of the wake vortex center above the ground
L = turbulence length scale
q_* = turbulence velocity scale
R_m = core radius equal to radius of maximum swirl velocity
t = time or age of the vortex
U = wind velocity
u = horizontal velocity of the vortex core
v_m = maximum swirl (tangential) velocity of the vortex
w = vertical velocity of the vortex core
X = horizontal position of the wake vortex center
z = vertical coordinate
α = angle the wind makes with the runway axis
Γ = circulation around the wake vortex
ρ = density of air
σ = vortex tilt parameter

Subscript
i = induced velocity

Introduction

THIS Note expands on a simple empirical model developed earlier for the transport and decay of aircraft trailing (wake) vortices in a turbulent atmosphere near the ground,[1] with potential applications to single and parallel runways. Our approach was similar to that of Greene,[2] but included the

Received Aug. 20, 1997; revision received March 17, 1998; accepted for publication March 24, 1998.

*Professor, Department of Aerospace Engineering Sciences, CB431. E-mail: kantha@colorado.edu. Senior Member AIAA.